Topology Qualifying Examination

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Instructions: Solve four out of the five problems. If you attempt more than four problems, indicate which four you want graded. You must justify your claims either by direct arguments or by referring to theorems you know.

Problem 1. Let X be a space built from a wedge of two circles by attaching three 2-cells using attaching maps specified by loops a^2, b^2 and $aba^{-1}b^{-1}$ (Here a and b are the loops given by the inclusion map of either circle.)

a) Calculate $\pi_1(X)$.

b) How many path connected covering spaces of X are there, up to equivalence? Of these, how many are normal covers? Justify your answers.

Problem 2. Let F_n denote the free group on n generators.

a) Use covering space theory to construct an explicit index two subgroup of F_3 which is isomorphic to F_5.

b) Use covering space theory to show that *any* index two subgroup of F_3 is isomorphic to F_5.

Problem 3. a) Calculate the homology groups of the torus $T^2 = S^1 \times S^1$.

b) Let $Z = T^2 \sqcup D^2/\sim$ be the space obtained from the torus and a disk by identifying the boundary circle of the disk ∂D^2 with the circle C that wraps three times around the first circle factor of the torus and once around the second i.e.

$$C = \{(e^{3i\theta}, e^{i\theta}) \in S^1 \times S^1 \mid \theta \in [0, 2\pi]\}.$$

Use the Mayer-Vietoris sequence to compute the homology groups of Z.

Problem 4. Let A denote the coordinate axes in $\mathbb{R}^2$ i.e.

$$A = \{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ OR } y = 0\}.$$

Show that there does NOT exist a homeomorphism from A to A which sends the origin $(0, 0)$ to the point $(1, 0)$.

Problem 5. Suppose that a space N has fundamental group, all of whose elements have odd order i.e. for every $g \in \pi_1(N)$ there exists k such that $g^{2k+1} = 1$. Show that any continuous map $f : N \to \mathbb{R}P^2$ induces the zero map on all reduced homology groups.